Cyclic Polygonal Designs with Block Size 3 and Joint Distance $\alpha = 2$

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Cyclic Polygonal Designs with Block Size 3 and Joint Distance $\lambda = 2$

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Polygonal designs are useful in survey sampling in terms of balanced sampling plans excluding contiguous units (BSECs) and balanced sampling plans excluding adjacent units (BSAs). In this article, the method of cyclic shifts has been used for the construction of cyclic polygonal designs (in terms of BSAs) with block size $k = 3$ and $\lambda = 1, 2, 3, 4, 6, 12$ for joint distance $\alpha = 2$ and 51 new designs for treatments $v \leq 100$ are given.

Keywords Cyclic BSA; Cyclic polygonal designs; Cyclic shifts; Distance between the units; Joint distance; PBIBD.

Mathematics Subject Classification 05B05; 62K05; 62K10.

1. Introduction

A study was conducted to investigate the species abundance, diversity, and richness of certain insects in a forest. For the selected plots, a net was placed under the trees, the trees were fogged with the insecticide and the insects of the species of interest that landed on the net were counted. This is an expensive and time-consuming procedure that can be applied only to a relatively small number of plots with small areas. In this study, it is expected that counts from neighboring plots would be very similar, and that “fogging” one plot could alter the responses in neighboring plots. There were certain regions in the forest where a low insect count was expected from all trees due to recent fires, and other regions, near a creek, where a relatively high count was expected. Therefore, a sampling plan that avoids the simultaneous selection of neighboring plots within a region was utilized (See and Song, 2002).
Hedayat et al. (1988a,b) first introduced balanced sampling plans excluding contiguous units (BSECs) in which the contiguous units do not appear together in a sample whereas all other pairs of units appear equally often. Stufken (1993) generalized the concept of BSECs to balanced sampling plans excluding adjacent units (BSAs) where all those adjacent pairs of units are excluded whose distance is less than or equal to \( z \), where \( z \) denotes the distance between the units. A cyclic polygonal designs (CPD) is actually a partially balanced incomplete block design with two-associates. The notation \( \text{BSEC}(v, k, \lambda) \) is used to denote a \( \text{CPD}(v, k, \lambda; 1) \) and the notation \( \text{BSA}(v, k, \lambda; z) \) is used to denote a \( \text{CPD}(v, k, \lambda; z) \).

**Definition 1.1.** A \( \text{CPD}(v, k, \lambda; z) \) for \( \{0, 1, \ldots, v-1\} \) treatments in \( b \) blocks of size \( k \) each \((k < v)\) and some \( \lambda \), is a binary block design in which a pair of treatments \((i, j)\) do not appear in any block if \( \delta_{(i,j)} \leq z \) and the pair of treatments \((i, j)\) appears together in \( \lambda \) blocks if \( \delta_{(i,j)} > z \), for all \((i, j) \in \{0, 1, \ldots, v-1\}\).

Hedayat et al. (1988b, p. 557) first introduced \( \text{CPD}(11, 3, 1; 2) \). Stufken (1993) gave the existence of \( \text{CPDs} \) (in terms of BSAs) for \( z \geq 2 \), and gave construction of \( \text{CPD}(20, 3, 2; 2) \) and \( \text{CPD}(15, 3, 3; 2) \). Wei (2002) suggested the use of Langford sequence for the existence and construction of \( \text{CPD}(v, 3, \lambda; 2) \)'s with \( \lambda = 1, 2 \) for arbitrary \( z \). Zhang and Chang (2005) gave the necessary conditions for the existence and construction of \( \text{CPD}(v, 3, \lambda; 2) \)'s and constructed \( \text{CPDs} \) with \( \lambda = 1, 2, 3, 4, 6, 12 \) for some \( v \). Mandal et al. (2008) used symmetrically repeated differences and linear programming approach and gave a catalog of \( \text{CPD}(v, 3, \lambda; 2) \)'s with \( \lambda = 1, 2, 3, 4, 6, 12 \) for some \( v \).

In this article, some new constructions of \( \text{CPD}(v, 3, \lambda; 2) \)'s are proposed for \( v \leq 100 \) and 51 new designs have been constructed whose parameters are given in Table 1. The method of cyclic shifts (which the authors have described in Iqbal et al., 2009) is used for the construction of these CPDs. In Sec. 2, some algorithms are given to search \( \text{CPD}(v, 3, \lambda; 2) \)'s with \( \lambda = 1, 2, 3, 4, 6, 12 \). The concluding remarks are given in Sec. 3.

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( v ) (Fractional and non-fractional CPDs using the method of cyclic shifts)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>{41, 47, 53, 59, 65, 71, 77, 83, 89, 95}</td>
</tr>
<tr>
<td>2</td>
<td>{36, 48, 60, 72, 84, 96}</td>
</tr>
<tr>
<td>3</td>
<td>{68, 80, 92}</td>
</tr>
<tr>
<td>4</td>
<td>{49, 55, 73, 79, 91, 97}</td>
</tr>
<tr>
<td>5</td>
<td>{63, 69, 75, 81, 87, 93, 99}</td>
</tr>
<tr>
<td>6</td>
<td>{50, 62, 74, 86, 98}</td>
</tr>
<tr>
<td>7</td>
<td>{66, 78, 90}</td>
</tr>
<tr>
<td>8</td>
<td>{52, 64, 76, 88, 100}</td>
</tr>
<tr>
<td>9</td>
<td>{34, 46, 58, 70, 82, 94}</td>
</tr>
</tbody>
</table>

where \( f \) stands for fractional (or smaller) CPDs.
2. Algorithms for the Construction of CPD(\(v; 3; \lambda; 2\))'s

Let \(S_j = \{q_{1j}, q_{2j}\}\) be the set of shifts, where \(\alpha \leq q_{1j}, q_{2j} \leq v - \alpha\), and \(q_{1j}\) and \(q_{2j}\) may be repeated any number of times. Then, \(S'_j = \{(q_{1j}, q_{2j}, (q_{1j} + q_{2j}) \mod v, v - q_{1j}, v - q_{2j}, v - (q_{1j} + q_{2j}) \mod v)\}\). Here, \(v - q_{1j}\) is complement of \(q_{1j}\).

A design will be CPD with \(\alpha = 2\), if:

(i) \(S'_j\) consists of \(3, \ldots, v - 3\) an equal number of times, say \(\lambda\), and

(ii) \((q_{1j} + q_{2j}) \mod v \neq 0\).

Some algorithms for the construction of CPDs with \(k = 3\) and \(\lambda = 1, 2, 3, 4, 6, 12\) for joint distance \(\alpha = 2\) are presented with the conditions that

(i) \(3 \leq q_{1j}, q_{2j} \leq v - 3\), and

(ii) \((q_{1j} + q_{2j}) \mod v \neq 0, 1, 2, v - 1, v - 2\).

**Algorithm 2.1.** A fractional CPD with \(k = 3\) and \(\lambda = 1\) for \(\alpha = 2\) can be constructed if \(v = 6i - 1; i (>2)\) is an integer, from the following \(i - 1\) sets of shifts:

\[
S_j = [q_{1j}, q_{2j}]; \quad j = 1, 2, \ldots, i - 1
\]

such that \(3, 4, \ldots, v - 3\) appear once among \(q_{1j}, q_{2j}\), \((q_{1j} + q_{2j}) \mod v\) and their complements.

**Algorithm 2.2A.** A fractional CPD with \(k = 3\) and \(\lambda = 2\) for \(\alpha = 2\) can be constructed if \(v = 12i; i (>1)\) is an integer, from the following \(4i - 2\) sets of shifts along with the fractional part \(S_{4i-1} = [v/3, v/3] (1/3)\):

\[
S_j = [q_{1j}, q_{2j}]; \quad j = 1, 2, \ldots, 4i - 2
\]

such that \(3, 4, \ldots, (v - 2)/2, (v + 2)/2, \ldots, v - 4, v - 3\) all appear twice but \((v/2)\) appears once among \(q_{1j}, q_{2j}\), \((q_{1j} + q_{2j}) \mod v\) and their complements.

**Algorithm 2.2B.** A CPD with \(k = 3\) and \(\lambda = 2\) for \(\alpha = 2\) can be constructed if \(v = 12i + 8; i (\geq 1)\) is an integer, from the following \(4i + 1\) sets of shifts:

\[
S_j = [q_{1j}, q_{2j}]; \quad j = 1, 2, \ldots, 4i + 1
\]

such that \(3, 4, \ldots, (v - 2)/2, (v + 2)/2, \ldots, v - 3\) appear twice but \((v/2)\) appears once among \(q_{1j}, q_{2j}\), \((q_{1j} + q_{2j}) \mod v\) and their complements.

**Algorithm 2.3A.** A CPD with \(k = 3\) and \(\lambda = 2\) for \(\alpha = 2\) can be constructed if \(v = 6i + 1; i (>2)\) is an integer, from the following \(3i - 2\) sets of shifts:

\[
S_j = [q_{1j}, q_{2j}]; \quad j = 1, 2, \ldots, 3i - 2
\]

such that \(3, 4, \ldots, v - 3\) appears thrice among \(q_{1j}, q_{2j}\), \((q_{1j} + q_{2j}) \mod v\) and their complements.
Algorithm 2.3B. A CPD with $k = 3$ and $\lambda = 3$ for $x = 2$ can be constructed if $v = 6i + 3; i (> 1)$ is an integer, from the following $3i - 1$ sets of shifts:

$$S_j = [q_{1j}, q_{2j}]; \ j = 1, 2, \ldots, 3i - 1$$

such that $3, 4, \ldots, v - 3$ appear thrice among $q_{1j}, q_{2j}, (q_{1j} + q_{2j})$ mod $v$ and their complements.

Algorithm 2.4A. A CPD with $k = 3$ and $\lambda = 4$ for $x = 2$ can be constructed if $v = 12i + 2; i (> 1)$ is an integer, from the following $8i - 2$ sets of shifts:

$$S_j = [q_{1j}, q_{2j}]; \ j = 1, 2, \ldots, 8i - 2$$

such that $3, 4, \ldots, (v - 2)/2, (v + 2)/2, \ldots, v - 4, v - 3$ appear four times but $(v/2)$ appears twice among $q_{1j}, q_{2j}, (q_{1j} + q_{2j})$ mod $v$ and their complements.

Algorithm 2.4B. A fractional CPD with $k = 3$ and $\lambda = 4$ for $x = 2$ can be constructed if $v = 12i + 6; i (\geq 1)$ is an integer, from the following $8i$ sets of shifts along with the fractional part $S_{9i-1} = [v/3, v/3] (2/3)$:

$$S_j = [q_{1j}, q_{2j}]; \ j = 1, 2, \ldots, 8i$$

such that $3, 4, \ldots, (v - 2)/2, (v + 2)/2, \ldots, v - 4, v - 3$ appear four times but $(v/2)$ appears twice among $q_{1j}, q_{2j}, (q_{1j} + q_{2j})$ mod $v$ and their complements.

Algorithm 2.5. A CPD with $k = 3$ and $\lambda = 6$ for $x = 2$ can be constructed if $v = 12i + 4; i (\geq 1)$ is an integer, from the following $12i - 1$ sets of shifts:

$$S_j = [q_{1j}, q_{2j}]; \ j = 1, 2, \ldots, 12i - 1$$

such that $3, 4, \ldots, (v - 2)/2, (v + 2)/2, \ldots, v - 3$ appear six times but $(v/2)$ appears thrice among $q_{1j}, q_{2j}, (q_{1j} + q_{2j})$ mod $v$ and their complements.

Algorithm 2.6. A CPD with $k = 3$ and $\lambda = 12$ for $x = 2$ can be constructed if $v = 12i + 10; i (\geq 1)$ is an integer, through the following $24i + 10$ sets of shifts:

$$S_j = [q_{1j}, q_{2j}]; \ j = 1, 2, \ldots, 24i + 10$$

such that $3, 4, \ldots, (v - 2)/2, (v + 2)/2, \ldots, v - 4, v - 3$ appear twelve times but $(v/2)$ appears six times among $q_{1j}, q_{2j}, (q_{1j} + q_{2j})$ mod $v$ and their complements.

3. Concluding Remarks

Hedayat et al. (1988b) first showed the construction of CPD(17, 3, 1; 2). Stufken (1993) also constructed CPD(20, 3, 2; 2) and CPD(15, 3, 3; 2). Wei (2002) suggested the use of Langford sequence for the construction of CPD($v, 3, \lambda; \alpha$)’s with $\lambda = 1, 2$ for arbitrary $\alpha$. Zhang and Chang (2005) used Langford sequence and extended Langford sequence, and constructed CPD($v, 3, \lambda; \beta$)’s with $\lambda = 1, 2, 3, 4, 6$ for some $v$. Zhang and Chang (2005) gave base blocks and triples for $v \in \{15, 16, 17, 18, 19, 20, 21, 23, 25, 26, 27, 28, 29, 30, 31, 32, 33, 37, 38, 39, 40, 42, 43, 44, 45, 51,
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54, 56, 57, 61, 67, 85}. Mandal et al. (2008) used symmetrically repeated differences and linear programming techniques for the construction of CPD(\(v, 3, \lambda; 2\))’s with \(\lambda = 1, 2, 3, 4, 6, 12\) for \(v \in \{16, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 29, 32, 33, 35\}\).

In this article, the authors have proposed 51 new CPD(\(v, 3, \lambda; 2\))’s with \(\lambda = 1, 2, 3, 4, 6, 12\) for \(v \leq 100\) treatments. Some of the designs are given in Appendix A and a complete catalog is available from first author’s webpage: http://www.iub.edu.pk/Teacher.php?dept_id=8&teacher_id=126 or by sending an email request to the address mtahir.stat@gmail.com.

Note: In fractional designs, certain sets of shifts produce design that are made up of complete replications of the smaller designs. Thus, we add one or more subsets of blocks contained within the design to one or more other subsets or complete designs. While a non-fractional design is constructed just by adding one or more sets of complete designs.

As an illustration, a fractional CPD(18, 3, 4; 2) can be constructed by combining together the blocks obtained from the nine sets of shifts [3, 5], [3, 7], [3, 6], [4, 4], [4, 5], [4, 7], [5, 6], and \([6, 6]\)’s, resulting into set of shifts, [3, 5](2) + [3, 6] + [3, 7] + [4, 4] + [4, 5] + [4, 7] + [5, 6] + [6, 6]’s. The first eight set of shifts produce complete design whereas the last set of shift [6, 6]’s gives fractional design. Let the complete design obtained from set of shifts [6, 6] is

<table>
<thead>
<tr>
<th>Blocks</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18</td>
</tr>
<tr>
<td>0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17</td>
</tr>
<tr>
<td>6 7 8 9 10 11 12 13 14 15 16 17 0 1 2 3 4 5</td>
</tr>
<tr>
<td>12 13 14 15 16 17 0 1 2 3 4 5 6 7 8 9 10 11</td>
</tr>
</tbody>
</table>

then if first 12 blocks out of 18 of the above design, that is \(\frac{12}{2}\) of [6, 6], are combined with the remaining design obtained from the other set of shifts, will result into a fractional design.

Appendix A

By using algorithms given in Sec. 2, only 22 out of 51 new CPD(\(v, 3, \lambda; 2\))’s are given below. These CPDs exist under the necessary condition \(\lambda = \frac{\beta(k-1)}{v-(2x+1)}\), where \(\beta = \frac{d_0}{6}\) denotes the number of shifts required for a CPD with \(x = 2\).

**CPD(\(v, 3, 1; 2\))’s with \(v \equiv 5 \pmod{6}\)**

\(v = 41 : [3, 4] + [4, 16] + [5, 13] + [6, 9] + [7, 12] + [10, 14]\)

\(v = 47 : [3, 12] + [4, 18] + [5, 11] + [6, 14] + [7, 17] + [8, 13] + [9, 10]\)

\(v = 53 : [3, 16] + [4, 23] + [5, 10] + [6, 18] + [7, 14] + [8, 12] + [9, 13] + [11, 17]\)

\(v = 59 : [3, 21] + [4, 23] + [5, 17] + [6, 20] + [7, 9] + [8, 11] + [10, 18] + [12, 13] + [14, 15]\)

\(v = 65 : [3, 15] + [4, 22] + [5, 8] + [6, 25] + [7, 16] + [9, 29] + [10, 14] + [11, 19] + [12, 21] + [17, 20]\)
Fractional CPD\((v, 3, 2; 2)\)'s with \(v \equiv 0 \pmod{12}\)

\[ v = 36 : [3, 9] + [3, 13] + [4, 11] + [4, 15] + [5, 8] + [5, 9] + [6, 8] + [6, 10] + [7, 10] + [7, 11] + [12, 12]\left(\frac{1}{3}\right)\]

\[ v = 48 : [3, 6] + [3, 17] + [4, 13] + [4, 16] + [5, 14] + [5, 18] + [6, 12] + [7, 12] + [7, 13] + [8, 14] + [8, 18] + [9, 15] + [10, 11](2) + [16, 16]\left(\frac{1}{3}\right)\]

\[ v = 60 : [3, 23](2) + [4, 8] + [4, 20] + [5, 10] + [5, 16] + [6, 11] + [6, 12] + [7, 18] + [7, 22] + [8, 19] + [9, 21] + [9, 22] + [10, 14] + [11, 17] + [13, 14] + [13, 15] + [16, 19] + [20, 20]\left(\frac{1}{3}\right)\]

CPD\((v, 3, 2; 2)\)'s with \(v \equiv 8 \pmod{12}\)

\[ v = 68 : [3, 26] + [3, 28] + [4, 13] + [4, 23] + [5, 16] + [5, 25] + [6, 13] + [6, 22] + [7, 17] + [7, 24] + [8, 12] + [8, 14] + [9, 18] + [9, 26] + [10, 15] + [10, 23] + [11, 19] + [11, 21] + [12, 20] + [14, 15] + [16, 18]\]

CPD\((v, 3, 3; 2)\)'s with \(v \equiv 1 \pmod{6}\)

\[ v = 49 : [3, 14] + [3, 15] + [3, 18] + [4, 12] + [4, 19](2) + [5, 9] + [5, 16] + [5, 17] + [6, 7] + [6, 10] + [6, 19] + [7, 11] + [7, 17] + [8, 12](2) + [8, 15] + [9, 11] + [9, 13] + [10, 11] + [10, 15] + [13, 14] + [15, 17]\]

\[ v = 55 : [3, 12] + [3, 21] + [3, 23] + [4, 16] + [4, 18] + [4, 22] + [5, 14] + [5, 16] + [5, 21] + [6, 7] + [6, 8] + [6, 19] + [7, 10] + [7, 18] + [8, 12] + [8, 14] + [9, 11] + [9, 15] + [9, 18] + [10, 13] + [10, 17] + [11, 13] + [11, 19] + [12, 16] + [15, 17]\]

CPD\((v, 3, 3; 2)\)'s with \(v \equiv 3 \pmod{6}\)

\[ v = 63 : [3, 21](2) + [3, 25] + [4, 10] + [4, 12] + [4, 25] + [5, 5] + [5, 9] + [6, 17](2) + [6, 25] + [7, 16] + [7, 19](2) + [8, 19] + [8, 22](2) + [9, 15] + [9, 18] + [10, 20] + [11, 16] + [11, 18] + [11, 20] + [12, 14] + [12, 17] + [12, 20] + [13, 15](2) + [20, 21]\]

\[ v = 69 : [3, 18](2) + [3, 25] + [4, 24] + [4, 26] + [4, 30] + [5, 15] + [5, 22] + [5, 30] + [6, 13] + [6, 23] + [6, 25] + [7, 18] + [7, 21] + [7, 26] + [8, 9] + [8, 14](2) + [9, 20] + [9, 26] + [10, 14] + [10, 23](2) + [11, 13] + [11, 16](2) + [12, 19](2) + [12, 20] + [13, 16] + [15, 17](2)\]
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CPD(v, 3, 4; 2)'s with \( v \equiv 2 \) (mod 12)

\[
v = 50 : [3, 11](2) + [3, 18](2) + [4, 18] + [4, 19](3) + [5, 10] + [5, 11] + [5, 12] + [5, 18] + [6, 15](2) + [6, 16](2) + [7, 12] + [7, 13](2) + [7, 17] + [8, 9](2) + [8, 12](2) + [9, 13] + [10, 14](2) + [10, 15] + [11, 13]
\]

\[
v = 62 : [3, 20](2) + [3, 26](2) + [4, 21](2) + [4, 24](2) + [5, 14](2) + [5, 22](2) + [6, 7] + [6, 10] + [6, 13] + [6, 17] + [7, 10] + [7, 13] + [7, 21] + [8, 15] + [8, 16](2) + [9, 13] + [9, 17](2) + [9, 19] + [10, 20] + [10, 21] + [11, 14](2) + [11, 18](2) + [12, 15](2) + [12, 18](2) + [15, 16]
\]

Fractional CPD(v, 3, 4; 2)'s with \( v \equiv 6 \) (mod 12)

\[
v = 66 : [3, 22](2) + [3, 23](2) + [4, 20](2) + [4, 25](2) + [5, 13] + [5, 26] + [5, 28](2) + [6, 7] + [6, 13] + [6, 24](2) + [7, 9] + [7, 10](2) + [8, 15](2) + [8, 18] + [8, 20] + [9, 11] + [9, 18](2) + [10, 17](2) + [11, 21](3) + [12, 16] + [12, 19](3) + [13, 21] + [14, 15](2) + [14, 16](2) + [22, 22]\left(\frac{2}{3}\right)
\]

CPD(v, 3, 6; 2)'s with \( v \equiv 4 \) (mod 12)

\[
v = 52 : [3, 17](3) + [3, 18](3) + [4, 7](2) + [4, 9](2) + [4, 15] + [4, 21] + [5, 16](2) + [5, 17](3) + [5, 19] + [6, 10] + [6, 18](3) + [6, 19](2) + [7, 10] + [7, 16](3) + [8, 11](2) + [8, 12](2) + [8, 15](2) + [9, 11] + [9, 13](3) + [10, 14](2) + [10, 17](2) + [11, 12] + [12, 14](3) + [13, 14]
\]

\[
v = 64 : [3, 11] + [3, 16](2) + [3, 22](3) + [4, 12] + [4, 17](2) + [4, 28](3) + [5, 22](3) + [5, 26](3) + [6, 14](3) + [6, 18] + [6, 19] + [6, 20] + [7, 8](2) + [7, 17] + [7, 19](3) + [8, 13] + [8, 15] + [8, 23](2) + [9, 15](3) + [9, 20](3) + [10, 13](2) + [10, 17](3) + [10, 23] + [11, 14](2) + [11, 18](3) + [12, 16](3) + [12, 18](2) + [13, 21](3)
\]

CPD(v, 3, 12; 2)'s with \( v \equiv 10 \) (mod 12)

\[
v = 34 : [3, 10](4) + [3, 11](5) + [3, 12](3) + [4, 7](2) + [4, 8](2) + [4, 10](2) + [4, 11](2) + [4, 12](4) + [5, 7](2) + [5, 8](4) + [5, 9](2) + [5, 10](2) + [5, 13](2) + [6, 7](2) + [6, 8](2) + [6, 9](4) + [6, 10](2) + [6, 11](2) + [7, 9](4) + [7, 10](2) + [8, 9](2) + [8, 11] + [8, 12]
\]

\[
v = 46 : [3, 7](2) + [3, 13](2) + [3, 14](4) + [3, 18](2) + [3, 19](2) + [4, 6](2) + [4, 10](2) + [4, 11](2) + [4, 16](2) + [4, 18](2) + [4, 19](2) + [5, 6](2)
\]
\[ v = 58 : [3, 19] + [3, 20](6) + [3, 24](5) + [4, 19] + [4, 21](5) + [4, 24](6) + [5, 12](6) + [5, 16](6) + [6, 9](2) + [6, 11](5) + [6, 16](4) + [6, 22] + [7, 13](6) + [7, 19](6) + [8, 8](2) + [8, 14](6) + [8, 17] + [8, 19] + [9, 14] + [9, 15] + [9, 18](6) + [9, 19](2) + [10, 11] + [10, 13](5) + [10, 15](6) + [12, 14](6) + [11, 18](6) + [13, 15] + [15, 15] \]

\[ v = 70 : [3, 14](4) + [3, 27](6) + [3, 29](2) + [4, 24](6) + [4, 29](6) + [5, 13](6) + [5, 23](6) + [6, 16](6) + [6, 27](6) + [7, 7] + [7, 12](5) + [7, 19] + [7, 22](4) + [8, 17](2) + [8, 18](6) + [8, 26](4) + [9, 21](6) + [9, 23](6) + [10, 15](10) + [10, 22](2) + [11, 13](6) + [11, 20](6) + [12, 14] + [12, 19](6) + [14, 21](6) + [15, 17](2) + [16, 20](6) + [17, 17](2) \]

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References


